

# COHERENCE CONVERTIBILITY FOR MIXED STATES

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**ABSTRACT.** In this paper, by providing a class of coherence measures in finite dimensional systems, a sufficient and necessary condition for the existence of coherence transformations that convert one probability distribution of any pure states into another one is obtained.

## 1. INTRODUCTION

Coherence is a fundamental aspect of quantum physics that encapsulates the defining features of the theory [1], from the superposition principle to quantum correlations. It is a key component in various quantum information and estimation protocols and is primarily accountable for the advantage offered by quantum tasks versus classical ones [2, 3]. It has been shown that a good definition of coherence does not only depend on the state of the system, but also depend on the a fixed basis for the quantum system [4]. So far, several themes of coherence have been considered such as witnessing coherence [5], catalytic coherence [6] and the thermodynamics of quantum coherence [7].

But, given a quantum state, how much the coherence does it contain? How to quantify the quantum coherence? There is no well-accepted efficient method to quantify the coherence in quantum system until recently. Baumgratz et al. [4] introduced a rigorous framework for quantification of coherence and proposed several measures of coherence, which are based on the well-behaved metrics including the  $l_p$ -norm, relative entropy, trace norm and fidelity. After then, the quantification of coherence stimulated a lot of further considerations (see [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]). Especially, a general method to derive a series of coherence measures from concave functions was given in [17] which plays a key role in this paper.

In quantum information science, the question what tasks may be accomplished using a given physical resource is of fundamental importance in many areas. It is well known that entanglement is a useful physical resource in many processes of quantum information processing. In order to perform some tasks, it is key to manipulate the entanglement under special conditions, namely allowing only local operations and classical communication (LOCC). The

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celebrated Nielsen theorem exposed the necessary and sufficient conditions for pure bipartite entanglement transformations [18]. Then Jonathan and Plenio [19] extended this result to the case that a pure bipartite state can be transformed into a probability distribution of pure states. Later, Li and Shi [20] gave necessary and sufficient conditions that a bipartite mixed state can be transformed into another mixed state by LOCC. Other related results can be found in [21, 22, 23] and the references therein.

For coherence, in [4], the authors proposed a question similar to entanglement: whether incoherent operations can introduce an order on the set of quantum states, i.e., whether, given any two states  $\rho$  and  $\sigma$ , either  $\rho$  can be transformed into  $\sigma$  or vice versa under incoherent operations. In [24], the authors gave an affirmative answer to this question for pure states by majorization condition.

In this letter, we are aimed to determine when a mixed state  $\rho$  can be transformed to a mixed state  $\sigma$  by incoherent operations. By constructing new classes of coherence measures using the method in [17], we partially answer this question raised in [4].

This paper is organized as follows. In Section 2, we introduce the concept of coherence measures and the approach how to construct general coherence measures. Based on these, two new classes of coherence measures are given, which are key for our main results. Section 3 is devoted to obtaining a necessary and sufficient condition that an ensemble can be transformed into another one by incoherent quantum operations. We summarize our results in Section 4.

## 2. COHERENCE MEASURES

**2.1. The construction of coherence measures.** In this paper, we always assume that  $H$  is a finite dimensional Hilbert space with  $\dim H = d$ . Let  $\mathcal{S}(H)$  be the space of all states on  $H$ . Fixing a particular basis  $\{|i\rangle\}_{i=1}^d$ , recall that a state  $\rho \in \mathcal{S}(H)$  is called incoherent if  $\rho$  is diagonal in the fixed basis, that is,  $\rho = \sum_{i=1}^d \lambda_i |i\rangle\langle i|$  with  $\lambda_i \geq 0$  and  $\sum_{i=1}^d \lambda_i = 1$ . Denote by  $\mathcal{I}$  the set of all incoherent quantum states in  $H$ . Quantum operations are specified by a finite set of Kraus operators  $\{K_n\}$  satisfying  $\sum_n K_n^\dagger K_n = I$ ,  $I$  is the identity operator on  $H$ . Quantum operations are incoherent (ICO) if they fulfil  $K_n \rho K_n^\dagger / \text{Tr}(K_n \rho K_n^\dagger) \in \mathcal{I}$  for all  $\rho \in \mathcal{I}$  and for all  $n$ . By [4], any proper measure of the coherence  $C$  must satisfy the following conditions:

$$(C1) \ C(\delta) = 0 \text{ for all } \delta \in \mathcal{I};$$

$$(C2a) \text{ Monotonicity under all incoherent operations (ICO) } \Phi: C(\rho) \geq C(\Phi(\rho)),$$

or (C2b) Monotonicity for average coherence under sub-selection based on measurements outcomes:  $C(\rho) \geq \sum_n p_n C(\rho_n)$  for all  $\{K_n\}$  with  $\sum_n K_n^\dagger K_n = I$  and  $K_n \mathcal{I} K_n^\dagger \subset \mathcal{I}$ , where  $\rho_n = \frac{K_n \rho K_n^\dagger}{p_n}$  and  $p_n = \text{Tr}(K_n \rho K_n^\dagger)$ ;

(C3) Non-increasing under mixing of quantum states:  $\sum_n p_n C(\rho_n) \geq C(\sum_n p_n \rho_n)$  for any set of states  $\{\rho_n\}$  and any  $p_n \geq 0$  with  $\sum_n p_n = 1$ .

Note that the conditions (C2b) and (C3) automatically imply the condition (C2a). The reason we listed all conditions above is that (similar to entanglement measures) there exist meaningful quantifiers of coherence which satisfy the conditions (C1), (C2a) and (C3), but for which the condition (C2b) is violated (see [8]).

Ref. [4] gave several coherence measures for finite dimensional systems, which are based on the well-behaved metrics such as the  $l_1$ -norm, relative entropy, trace norm and fidelity. Recently, Du, Bai and Qi [17] gave a general approach by convex roof to construct coherence measures for finite dimensional systems. The relative entropy coherence measure and  $l_1$  norm coherence measure can be derived from the approach.

Let  $\Omega = \{\mathbf{x} = (x_1, x_2, \dots, x_d)^t \mid \sum_{i=1}^d x_i = 1 \text{ and } x_i \geq 0\}$ , where  $(x_1, x_2, \dots, x_d)^t$  denotes the transpose of row vector  $(x_1, x_2, \dots, x_d)$ . It is not difficult to check that  $\Omega$  is a closed set in  $\mathbb{R}^d$ . Assume that  $f : \Omega \mapsto \mathbb{R}^+$  is any nonnegative function satisfying the following conditions:

- (i)  $f((1, 0, \dots, 0)^t) = 0$ ;
- (ii)  $f$  is invariant under any permutation transformation  $P_\pi$  (here,  $\pi$  is a permutation of  $\{1, 2, \dots, d\}$  and  $P_\pi$  is the permutation matrix corresponding to  $\pi$ ):  $f(P_\pi \mathbf{x}) = f(\mathbf{x})$  for every  $\mathbf{x} \in \Omega$ ;
- (iii)  $f$  is concave:  $f(\lambda \mathbf{x} + (1 - \lambda) \mathbf{y}) \geq \lambda f(\mathbf{x}) + (1 - \lambda) f(\mathbf{y})$  for all  $\lambda \in [0, 1]$  and all  $\mathbf{x}, \mathbf{y} \in \Omega$ .

For any pure state  $|\psi\rangle\langle\psi|$  with  $|\psi\rangle = \sum_{i=1}^d \psi_i |i\rangle \in H$ , define

$$C_f(|\psi\rangle) = f((|\psi_1|^2, |\psi_2|^2, \dots, |\psi_d|^2)^t).$$

For any mixed state  $\rho$ , define

$$C_f(\rho) = \min_{p_i, |\psi_i\rangle} \left\{ \sum_i p_i C_f(|\psi_i\rangle) : \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \right\}. \quad (1)$$

Then such  $C_f$  is a coherence measure satisfying (C2b) by [17, Theorem 1].

**2.2. New coherence measures.** For any vector  $\mathbf{x} = (x_1, x_2, \dots, x_d)^t \in \Omega$ , let  $\pi$  be the permutation of  $\{1, 2, \dots, d\}$  such that  $x_{\pi(1)} \geq x_{\pi(2)} \geq \dots \geq x_{\pi(d)}$ . Define a function  $f_l : \Omega \rightarrow \mathbb{R}^+$  by

$$f_l(\mathbf{x}) = \sum_{i=l}^d x_{\pi(i)}, \quad l = 2, 3, \dots, d.$$

It is easily checked that  $f_l$  ( $2 \leq l \leq d$ ) fulfills the conditions (i) and (iii). Also note that any permutation does not change elements of the vector. So (ii) is also satisfied by  $f_l$  for  $2 \leq l \leq d$ . Thus, by [17], for each  $l \in \{2, 3, \dots, d\}$ ,  $C_{f_l}$  is a coherence measure.

More generally, for any vector  $\mathbf{x} = (x_1, x_2, \dots, x_d)^t \in \Omega$ , we can define another function  $g_{l,k} : \Omega \rightarrow \mathbb{R}^+$  by

$$g_{l,k}(\mathbf{x}) = \sum_{i=l}^d \frac{x_{\pi(i)}}{k} \wedge 1, \quad l = 2, \dots, d, \quad \forall k \in (0, 1],$$

here  $\frac{x_{\pi(i)}}{k} \wedge 1$  denotes the minimal value of  $\frac{x_{\pi(i)}}{k}$  and 1. It is clear that each  $g_{l,k}$  fulfills the conditions (i) and (ii). Since the following equation

$$(\lambda a + (1 - \lambda)b) \wedge 1 \geq \lambda(a \wedge 1) + (1 - \lambda)(b \wedge 1)$$

holds for any  $a, b \geq 0$  and any  $\lambda \in [0, 1]$ , one can show that  $g_{l,k}$  is concave, that is,  $g_{l,k}$  satisfies the condition (iii) for  $l \in \{2, 3, \dots, d\}$  and for any  $k \in (0, 1]$ . So, by [17] again, each  $C_{g_{l,k}}$  is also a coherence measure. Particularly, if  $k = 1$ , then  $C_{g_{l,1}} = C_{f_l}$ .

### 3. CONVERTIBILITY BETWEEN MIXED STATES

**3.1. Convertibility between pure states.** In [4], the authors proposed a question: whether incoherent operations can introduce an order on the set of quantum states, i.e., whether, given two states  $\rho$  and  $\sigma$ , either  $\rho$  can be transformed into  $\sigma$  or vice versa. In [24], the authors gave an affirmative answer by majorization to the question for pure states, and they proved that, *for any unit vectors  $|\phi\rangle = \sum_{i=1}^d \phi_i|i\rangle, |\psi\rangle = \sum_{i=1}^d \psi_i|i\rangle \in H$ ,  $|\phi\rangle\langle\phi|$  can be transformed to  $|\psi\rangle\langle\psi|$  by using an incoherent operation if and only if  $C_{f_l}(|\phi\rangle\langle\phi|) \geq C_{f_l}(|\psi\rangle\langle\psi|)$  for all  $2 \leq l \leq d$ .* Here, the condition  $C_{f_l}(|\phi\rangle\langle\phi|) \geq C_{f_l}(|\psi\rangle\langle\psi|)$  for all  $2 \leq l \leq d$  is in fact equivalent to the condition  $(|\phi_1|^2, |\phi_2|^2, \dots, |\phi_d|^2)^t \prec (|\psi_1|^2, |\psi_2|^2, \dots, |\psi_d|^2)^t$ , that is,  $\sum_{i=1}^m |\phi_i|^2 \leq \sum_{i=1}^m |\psi_i|^2$  for  $1 \leq m \leq d - 1$ .

However, in practical application, people often need to deal with mixed states rather than pure ones. Thus, a natural problem is raised: whether or not majorization is a suitable tool for transformation from one mixed state into another one. If it is not true, what is the condition that a mixed state  $\rho$  can be transformed into another mixed state  $\sigma$  by using an incoherent quantum operation, that is,  $\rho \xrightarrow{\text{ICO}} \sigma$ . This will be the purpose of the next subsection.

**3.2. Convertibility between mixed states.** In this subsection, we will discuss the coherence convertibility between any mixed states.

We first consider the coherence transformation between any two ensembles. Let  $D_1 = \{p_j, |\phi_j\rangle\}_{j=1}^m$  and  $D_2 = \{q_i, |\psi_i\rangle\}_{i=1}^n$  be any two ensembles. Assume that, for each  $j \in$

$\{1, \dots, m\}$ , there exists an ICO  $\Phi_j$  which outputs the pure states  $|\psi_i\rangle$  with conditional probability  $t_{ji}$  for all the possible outcome states, that is,  $\Phi_j(|\phi_j\rangle\langle\phi_j|) = \sum_{i=1}^n t_{ji} |\psi_i\rangle\langle\psi_i|$  with  $\sum_{i=1}^n t_{ji} = 1$ . Here

$$t_{ji} = \text{tr}(K_l^{(j)} |\phi_j\rangle\langle\phi_j| K_l^{(j)\dagger}), \quad |\psi_i\rangle = \frac{1}{\sqrt{t_{ji}}} K_l^{(j)} |\phi_j\rangle$$

and  $K_l^{(j)}$  are Kraus operators of  $\Phi_j$ . Thus, the transformation  $\Phi \equiv (\Phi_1, \dots, \Phi_m)$ , defined between the ensembles  $D_1$  and  $D_2$ , outputs the states  $|\psi_i\rangle$  with probability  $q_i = \sum_{j=1}^m p_j t_{ji}$ , that is,

$$\Phi\left(\sum_{j=1}^m p_j |\phi_j\rangle\langle\phi_j|\right) = \sum_{i=1}^n q_i |\psi_i\rangle\langle\psi_i|.$$

If such  $\Phi$  exists, we say that  $D_1$  can be transformed into  $D_2$  by an incoherent operation, that is,  $D_1 = \{p_j, |\phi_j\rangle\} \xrightarrow{\text{ICO}} \{q_i, |\psi_i\rangle\} = D_2$ . Particularly, if  $m = 1$ , then  $|\phi_1\rangle \xrightarrow{\text{ICO}} \{q_i, |\psi_i\rangle\}$  implies that there exists an ICO  $\Phi_1$  which outputs pure states  $|\psi_i\rangle$  with probability  $q_i$ .

The following first result gives a necessary and sufficient condition of coherence transformations between pure states and any ensembles.

**Theorem 1.** *For any pure state  $|\phi\rangle\langle\phi|$  and any ensemble  $\{p_j, |\psi_j\rangle\}_{j=1}^m$ ,  $|\phi\rangle\langle\phi| \xrightarrow{\text{ICO}} \{p_j, |\psi_j\rangle\}_{j=1}^m$  if and only if  $C_{f_l}(|\phi\rangle) \geq \sum_{j=1}^m p_j C_{f_l}(|\psi_j\rangle)$  for  $l = 2, \dots, d$ .*

**Proof.** “ $\Rightarrow$ ”: Assume that there exists some incoherent quantum operation  $\Phi$  such that  $\Phi(|\phi\rangle\langle\phi|) = \sum_{j=1}^m p_j |\psi_j\rangle\langle\psi_j|$ . Since  $C_{f_l}$  is a coherence measure by Section 2, (C2b) implies  $C_{f_l}(|\phi\rangle) \geq \sum_{j=1}^m p_j C_{f_l}(|\psi_j\rangle)$  for  $l = 2, \dots, d$ .

“ $\Leftarrow$ ”: Assume that  $C_{f_l}(|\phi\rangle) \geq \sum_{j=1}^m p_j C_{f_l}(|\psi_j\rangle)$  for  $l = 2, \dots, d$ . Write  $|\phi\rangle = \sum_{i=1}^d \phi_i |i\rangle$  and  $|\psi_j\rangle = \sum_{i=1}^d \psi_{ji} |i\rangle$ . For the convenience, without loss of generality, we may require that the coefficients of  $|\phi\rangle$  and  $|\psi_i\rangle$  are all in the descending order. Thus, by the definition of  $C_{f_l}$ , we have

$$\begin{aligned} \sum_{i=l}^d |\phi_i|^2 &= C_{f_l}(|\phi\rangle) \geq \sum_{j=1}^m p_j C_{f_l}(|\psi_j\rangle) \\ &= \sum_{j=1}^m p_j \sum_{i=l}^d |\psi_{ji}|^2 \\ &= \sum_{i=l}^d \sum_{j=1}^m |\sqrt{p_j} \psi_{ji}|^2. \end{aligned} \tag{2}$$

Define a vector  $|\eta\rangle \in H$  by

$$|\eta\rangle = \sum_{i=1}^d \sqrt{\sum_{j=1}^m |\sqrt{p_j} \psi_{ji}|^2} |i\rangle = \sum_{i=1}^d \eta_i |i\rangle.$$

Note that

$$\sum_{i=1}^d |\eta_i|^2 = \sum_{i=1}^d \sum_{j=1}^m |\sqrt{p_j} \psi_{ji}|^2 = \sum_{j=1}^m p_j \left( \sum_{i=1}^d |\psi_{ji}|^2 \right) = \sum_{j=1}^m p_j = 1.$$

So  $|\eta\rangle\langle\eta|$  is a pure state. Moreover, Eq.(2) implies

$$C_{f_l}(|\phi\rangle) \geq C_{f_l}(|\eta\rangle), \quad 2 \leq l \leq d.$$

By [24], there exists an ICO  $\Phi_1$  such that

$$\Phi_1(|\phi\rangle\langle\phi|) = |\eta\rangle\langle\eta|. \quad (3)$$

Next, for any  $1 \leq j \leq m$ , define

$$A_j = \sum_{i=1}^d \frac{\sqrt{p_j} \psi_{ji}}{\eta_i} |i\rangle\langle i|.$$

It is easy to check that  $\sum_{j=1}^m A_j^\dagger A_j = I_d$  and  $A_j \mathcal{I} A_j^\dagger \subset \mathcal{I}$  for each  $j$ . So the map  $\Phi_2$  defined by  $\Phi(\cdot) = \sum_{j=1}^m A_j(\cdot) A_j^\dagger$  is an ICO. In addition, a simple calculation yields

$$A_j |\eta\rangle = p_j |\psi_j\rangle. \quad (4)$$

Define a new map  $\Phi$  by  $\Phi = \Phi_2 \circ \Phi_1$ . It is obvious that a composition of any two ICOs is still an ICO. So  $\Phi$  is an ICO; moreover, by Eqs.(3)-(4),  $\Phi$  realizes the required transformation.  $\square$

However, if the pure state  $|\phi\rangle\langle\phi|$  in Theorem 1 is replaced by an ensemble, then the coherence measure  $C_{f_l}$  is not enough to characterize the coherence transformations between the ensembles. In this case, more coherence measures are needed.

**Theorem 2.** Assume that  $D_1 = \{p_j, |\phi_j\rangle\}_{j=1}^m$  and  $D_2 = \{q_i, |\psi_i\rangle\}_{i=1}^n$  are any two ensembles. Then  $D_1 \xrightarrow{\text{ICO}} D_2$  if and only if  $\sum_{j=1}^m p_j C_{g_{l,k}}(|\phi_j\rangle) \geq \sum_{i=1}^n q_i C_{g_{l,k}}(|\psi_i\rangle)$  for  $l = 2, \dots, d$  and  $k \in (0, 1]$ .

To prove the theorem, the following lemma is needed.

**Lemma 3.** ([20]) Assume that  $\{p_i\}_{i=1}^n \subset [0, 1]$  and  $a \in [0, 1]$ . Then a set of  $\{a'_i\}_{i=1}^n$  satisfies  $\sum_{i=1}^n p_i a'_i \geq a$  if and only if there exists a set of  $\{a_i\}_{i=1}^n$  such that  $a_i \leq a'_i$  for all  $1 \leq i \leq n$  and  $\sum_{i=1}^n p_i a_i = a$ .

**Proof of Theorem 2.** For the “only if” part, if  $D_1 \xrightarrow{\text{ICO}} D_2$ , by the definition, for each  $j$ , there exists an ICO  $\Phi_j : M_d \rightarrow M_d$  such that  $\Phi_j(|\phi_j\rangle\langle\phi_j|) = \sum_{i=1}^n t_{ji} |\psi_i\rangle\langle\psi_i|$  with  $\sum_{i=1}^n t_{ji} = 1$  and  $q_i = \sum_{j=1}^m p_j t_{ji}$ . Since  $C_{g_{l,k}}$  is a coherence measure by Section 2, (C2b) implies  $C_{g_{l,k}}(|\phi_j\rangle) \geq \sum_{i=1}^n t_{ji} C_{g_{l,k}}(|\psi_i\rangle)$ , and so

$$\sum_{j=1}^m p_j C_{g_{l,k}}(|\phi_j\rangle) \geq \sum_{j=1}^m p_j \sum_{i=1}^n t_{ji} C_{g_{l,k}}(|\psi_i\rangle) = \sum_{i=1}^n q_i C_{g_{l,k}}(|\psi_i\rangle).$$

For the “if” part, assume that

$$\sum_{j=1}^m p_j C_{g_{l,k}}(|\phi_j\rangle) \geq \sum_{i=1}^n q_i C_{g_{l,k}}(|\psi_i\rangle) \quad \text{holds for } l = 2, \dots, d \text{ and } k \in (0, 1]. \quad (5)$$

Here, we only give the detailed proof for the case  $d = 3$  and  $m = n = 2$  in Eq.(5). For higher dimensional cases and larger  $m, n$ , the proof is similar.

Write  $|\psi_j\rangle = \sum_{i=1}^3 \psi_{ji}|i\rangle$  and  $|\phi_j\rangle = \sum_{i=1}^3 \phi_{ji}|i\rangle$ ,  $j = 1, 2$ . For the convenience, without loss of generality, we can require  $|\psi_{i1}|^2 \geq |\psi_{i2}|^2 \geq |\psi_{i3}|^2$  and  $|\phi_{i1}|^2 \geq |\phi_{i2}|^2 \geq |\phi_{i3}|^2$  for  $j = 1, 2$ . Then Eq.(5) implies

$$p_1\left(\frac{|\psi_{13}|^2}{k} \wedge 1\right) + p_2\left(\frac{|\psi_{23}|^2}{k} \wedge 1\right) \geq q_1\left(\frac{|\phi_{13}|^2}{k} \wedge 1\right) + q_2\left(\frac{|\phi_{23}|^2}{k} \wedge 1\right) \quad (6)$$

and

$$\begin{aligned} & p_1\left(\frac{|\psi_{12}|^2 + |\psi_{13}|^2}{k} \wedge 1\right) + p_2\left(\frac{|\psi_{22}|^2 + |\psi_{23}|^2}{k} \wedge 1\right) \\ & \geq q_1\left(\frac{|\phi_{12}|^2 + |\phi_{13}|^2}{k} \wedge 1\right) + q_2\left(\frac{|\phi_{22}|^2 + |\phi_{23}|^2}{k} \wedge 1\right), \end{aligned} \quad (7)$$

where  $k \in (0, 1]$  is arbitrary.

Assume that we have proved that there exists a set of  $\{t_{ij}\}_{i,j=1,2} \subseteq [0, 1]$  such that

$$C_{f_l}(|\psi_i\rangle) \geq t_{i1}C_{f_l}(|\phi_1\rangle) + t_{i2}C_{f_l}(|\phi_2\rangle) \quad (8)$$

holds for  $l = 2, 3$  and  $i = 1, 2$ , where  $\{t_{ij}\}_{i,j=1,2}$  satisfy the conditions:

$$\begin{cases} t_{11} + t_{12} = t_{21} + t_{22} = 1, \\ q_1 = p_1 t_{11} + p_2 t_{21}. \end{cases} \quad (9)$$

Then, by Theorem 1, for each  $i$ , there exists an ICO  $\Phi_i$  such that

$$\Phi_i(|\psi_i\rangle\langle\psi_i|) = t_{i1}|\phi_1\rangle\langle\phi_1| + t_{i2}|\phi_2\rangle\langle\phi_2|.$$

It follows from the definition in the first paragraph in Section 3.2 that  $\{p_1, p_2, |\psi_1\rangle, |\psi_2\rangle\} \xrightarrow{\text{ICO}} \{q_1, q_2, |\phi_1\rangle, |\phi_2\rangle\}$ , and so the “if” part holds.

Thus, to show the sufficiency, we only need to check the existence of  $\{t_{ij}\}$  satisfying Ineq.(8) and Eq.(9). Note that, by the definitions of  $C_{f_l}$ , Ineq.(8) implies

$$\begin{cases} (a1): & |\psi_{13}|^2 \geq t_{11}|\phi_{13}|^2 + (1 - t_{11})|\phi_{23}|^2, \\ (a2): & |\psi_{12}|^2 + |\psi_{13}|^2 \geq t_{11}(|\phi_{12}|^2 + |\phi_{13}|^2) \\ & + (1 - t_{11})(|\phi_{22}|^2 + |\phi_{23}|^2), \\ (b1): & |\psi_{23}|^2 \geq t_{21}|\phi_{13}|^2 + (1 - t_{21})|\phi_{23}|^2, \\ (b2): & |\psi_{22}|^2 + |\psi_{23}|^2 \geq t_{21}(|\phi_{12}|^2 + |\phi_{13}|^2) \\ & + (1 - t_{21})(|\phi_{22}|^2 + |\phi_{23}|^2), \end{cases} \quad (10)$$

which are respectively equivalent to the following inequalities:

$$\left\{ \begin{array}{l} (a1') : t_{11} \leq \frac{|\psi_{13}|^2 - |\phi_{23}|^2}{|\phi_{13}|^2 - |\phi_{23}|^2} \triangleq S_1, \\ (a2') : t_{11} \leq \frac{|\psi_{12}|^2 + |\psi_{13}|^2 - |\phi_{22}|^2 - |\phi_{23}|^2}{|\phi_{12}|^2 + |\phi_{13}|^2 - |\phi_{22}|^2 - |\phi_{23}|^2} \triangleq T_1, \\ (b1') : t_{21} \leq \frac{|\psi_{23}|^2 - |\phi_{23}|^2}{|\phi_{13}|^2 - |\phi_{23}|^2} \triangleq S_2, \\ (b2') : t_{21} \leq \frac{|\psi_{22}|^2 + |\psi_{23}|^2 - |\phi_{22}|^2 - |\phi_{23}|^2}{|\phi_{12}|^2 + |\phi_{13}|^2 - |\phi_{22}|^2 - |\phi_{23}|^2} \triangleq T_2. \end{array} \right. \quad (11)$$

So, in the rest of the paper, our goal is to prove the existence of  $\{t_{ij}\}$  satisfying Eq.(9) and Ineq.(10) (or (11)) by using Ineqs.(6)-(7) and Lemma 3.

We will complete it by considering several cases.

**Case 1.**  $C_{f_l}(|\psi_1\rangle) \geq C_{f_l}(|\psi_2\rangle)$  and  $C_{f_l}(|\phi_1\rangle) \geq C_{f_l}(|\phi_2\rangle)$  for  $l = 2, 3$ .

This case forces to  $C_{f_l}(|\psi_2\rangle) \geq C_{f_l}(|\phi_2\rangle)$  for  $l = 2, 3$ . Otherwise, there exists some  $l \in \{2, 3\}$ , without loss of generality, assume  $l = 3$ , such that  $C_{f_3}(|\psi_2\rangle) < C_{f_3}(|\phi_2\rangle)$ , that is,  $|\psi_{23}|^2 < |\phi_{23}|^2$ . Note that, the assumptions  $C_{f_3}(|\psi_1\rangle) \geq C_{f_3}(|\psi_2\rangle)$  and  $C_{f_3}(|\phi_1\rangle) \geq C_{f_3}(|\phi_2\rangle)$  respectively implies  $|\psi_{13}|^2 \geq |\psi_{23}|^2$  and  $|\phi_{13}|^2 \geq |\phi_{23}|^2$ . Then, taking  $k = |\phi_{23}|^2$  in Ineq.(6), one gets

$$1 = p_1 + p_2 > p_1 \left( \frac{|\psi_{13}|^2}{|\phi_{23}|^2} \wedge 1 \right) + p_2 \left( \frac{|\psi_{23}|^2}{|\phi_{23}|^2} \wedge 1 \right) \geq q_1 + q_2 = 1,$$

a contradiction. So for  $l = 2, 3$ , we have

$$\left\{ \begin{array}{l} C_{f_l}(|\psi_1\rangle) \geq C_{f_l}(|\psi_2\rangle) \geq C_{f_l}(|\phi_2\rangle), \\ C_{f_l}(|\phi_1\rangle) \geq C_{f_l}(|\phi_2\rangle). \end{array} \right.$$

**Subcase 1.1.**  $C_{f_l}(|\psi_1\rangle) \geq C_{f_l}(|\psi_2\rangle) \geq C_{f_l}(|\phi_1\rangle) \geq C_{f_l}(|\phi_2\rangle)$ ,  $l = 2, 3$ .

It is easy to check that

$$C_{f_l}(|\psi_i\rangle) \geq tC_{f_l}(|\phi_1\rangle) + (1-t)C_{f_l}(|\phi_2\rangle)$$

for  $i = 1, 2$  and any  $t \in [0, 1]$ . This implies that Ineq.(10) holds for any  $t_{ij}$ . So we only need to choose suitable  $t_{ij}$  such that Eq.(9) holds. Obviously, such  $t_{ij}$  are existent.

**Subcase 1.2.**  $C_{f_l}(|\psi_1\rangle) \geq C_{f_l}(|\phi_1\rangle) \geq C_{f_l}(|\psi_2\rangle) \geq C_{f_l}(|\phi_2\rangle)$ ,  $l = 2, 3$ .

Clearly,  $S_1, T_1 \geq 1$  in this case. So Ineq.(11)(a1')-(a2') hold for any  $t_{11} \in [0, 1]$ .

For  $|\psi_2\rangle$ , by Lemma 3, there exist  $t_{11}, t_{21} \in [0, 1]$  such that Eq.(9) and Ineq.(11)(b1') hold if and only if

$$p_1 + p_2 \frac{|\psi_{23}|^2 - |\phi_{23}|^2}{|\phi_{13}|^2 - |\phi_{23}|^2} \geq q_1; \quad (12)$$

and there exist  $t_{11}, t_{21} \in [0, 1]$  such that Eq.(9) and Ineq.(11)(b2') hold if and only if

$$p_1 + p_2 \frac{|\psi_{22}|^2 + |\psi_{23}|^2 - |\phi_{22}|^2 - |\phi_{23}|^2}{|\phi_{12}|^2 + |\phi_{13}|^2 - |\phi_{22}|^2 - |\phi_{23}|^2} \geq q_1. \quad (13)$$



By taking  $k = |\phi_{13}|^2$  and  $k = |\phi_{12}|^2 + |\phi_{13}|^2$  in Ineqs.(6)-(7), respectively, and by the assumption, one obtains

$$p_1|\phi_{13}|^2 + p_2|\psi_{23}|^2 \geq q_1|\phi_{13}|^2 + q_2|\phi_{23}|^2 \quad (14)$$

and

$$\begin{aligned} & p_1(|\phi_{12}|^2 + |\phi_{13}|^2) + p_2(|\psi_{22}|^2 + |\psi_{23}|^2) \\ \geq & q_1(|\phi_{12}|^2 + |\phi_{13}|^2) + q_2(|\phi_{22}|^2 + |\phi_{23}|^2). \end{aligned} \quad (15)$$

A simple calculation gets Ineq.(12) $\Leftrightarrow$ Ineq.(14) and Ineq.(13) $\Leftrightarrow$ Ineq.(15).

Now, by taking  $t_{21} \leq \min\{S_0, T_0\}$ , the above discussion guarantees that there exist  $t_{11}$  and  $t_{21}$  such that Eq.(9) and Ineq.(11) can be satisfied.

**Subcase 1.3.**  $C_{f_l}(|\phi_1\rangle) \geq C_{f_l}(|\psi_1\rangle) \geq C_{f_l}(|\psi_2\rangle) \geq C_{f_l}(|\phi_2\rangle)$ ,  $l = 2, 3$ .

For  $|\psi_1\rangle$  and  $|\psi_2\rangle$ , by Lemma 3, there exist  $t_{11}, t_{21} \in [0, 1]$  such that Eq.(9), Ineqs.(11)(a1') and (b1') hold if and only if

$$p_1 \frac{|\psi_{13}|^2 - |\phi_{23}|^2}{|\phi_{13}|^2 - |\phi_{23}|^2} + p_2 \frac{|\psi_{23}|^2 - |\phi_{23}|^2}{|\phi_{13}|^2 - |\phi_{23}|^2} \geq q_1,$$

that is,

$$p_1|\psi_{13}|^2 + p_2|\psi_{23}|^2 \geq q_1|\phi_{13}|^2 + q_2|\phi_{23}|^2; \quad (16)$$

and there exist  $t_{11}, t_{21} \in [0, 1]$  such that Eq.(9) and Ineqs.(11)(a2') and (b2') hold if and only if

$$p_1 \frac{|\psi_{12}|^2 + |\psi_{13}|^2 - |\phi_{22}|^2 - |\phi_{23}|^2}{|\phi_{12}|^2 + |\phi_{13}|^2 - |\phi_{22}|^2 - |\phi_{23}|^2} + p_2 \frac{|\psi_{22}|^2 + |\psi_{23}|^2 - |\phi_{22}|^2 - |\phi_{23}|^2}{|\phi_{12}|^2 + |\phi_{13}|^2 - |\phi_{22}|^2 - |\phi_{23}|^2} \geq q_1,$$

that is,

$$\begin{aligned} & p_1(|\psi_{12}|^2 + |\psi_{13}|^2) + p_2(|\psi_{22}|^2 + |\psi_{23}|^2) \\ \geq & q_1(|\phi_{12}|^2 + |\phi_{13}|^2) + q_2(|\phi_{22}|^2 + |\phi_{23}|^2). \end{aligned} \quad (17)$$

By taking  $k = 1$  in Ineqs.(6)-(7), one can get Ineqs.(16) and (17). So, by taking  $t_{11} \leq \min\{S_1, T_1\}$  and  $t_{21} \leq \min\{S_2, T_2\}$ , the above discussion guarantees that there exist  $t_{11}$  and  $t_{21}$  such that Eq.(9) and Ineq.(11) can be satisfied.

**Subcase 1.4.**  $C_{f_i}(|\psi_1\rangle) \geq C_{f_i}(|\phi_1\rangle)$  and  $C_{f_j}(|\psi_1\rangle) < C_{f_j}(|\phi_1\rangle)$  for  $i \neq j \in \{2, 3\}$ .

In this case, we have either

$$\begin{cases} C_{f_i}(|\psi_1\rangle) \geq C_{f_i}(|\psi_2\rangle) \geq C_{f_i}(|\phi_2\rangle), \\ C_{f_i}(|\phi_1\rangle) \geq C_{f_i}(|\psi_2\rangle) \geq C_{f_i}(|\phi_2\rangle); \end{cases} \quad (18)$$

or

$$\left\{ \begin{array}{l} C_{f_i}(|\psi_1\rangle) \geq C_{f_i}(|\psi_2\rangle) \geq C_{f_i}(|\phi_2\rangle), \\ C_{f_i}(|\phi_1\rangle) \geq C_{f_i}(|\phi_2\rangle), \\ C_{f_i}(|\psi_2\rangle) \geq C_{f_i}(|\phi_1\rangle), \\ C_{f_j}(|\psi_2\rangle) < C_{f_j}(|\phi_1\rangle). \end{array} \right. \quad (19)$$

Without loss of generality, assume that  $i = 2$  and  $j = 3$ . We only deal with the case (18). For the case (19), the proof is similar to that of Eq.(18).

Obviously,  $T_1 \geq 1$ , and so Ineq.(11)(a2') holds for any  $t_{11}$ . Thus, there exist  $t_{11}, t_{21} \in [0, 1]$  such that Eq.(9), Ineq.(11)(a1') and (b1') hold if and only if Ineq.(16) holds; there exist  $t_{11}, t_{21} \in [0, 1]$  such that Eq.(9) and Ineq.(11)(b2') hold if and only if Ineq.(13) holds. By taking  $k = |\phi_{13}|^2$  and  $k = |\phi_{12}|^2 + |\phi_{13}|^2$  in Ineqs.(6)-(7), respectively, Ineq.(16) and Ineq.(13) hold. Hence suitable  $t_{11}$  and  $t_{21}$  satisfying Eq.(9) and Ineq.(11) exist.

**Subcase 1.5.**  $C_{f_i}(|\psi_2\rangle) \geq C_{f_i}(|\phi_1\rangle)$  and  $C_{f_j}(|\psi_2\rangle) < C_{f_j}(|\phi_1\rangle)$  for  $i \neq j \in \{2, 3\}$ .

The proof is similar to that of Subcase 1.4.

**Case 2.**  $C_{f_i}(|\psi_1\rangle) \geq C_{f_i}(|\psi_2\rangle)$  and  $C_{f_j}(|\psi_1\rangle) < C_{f_j}(|\psi_2\rangle)$  for  $i \neq j \in \{2, 3\}$ .

**Case 3.**  $C_{f_i}(|\phi_1\rangle) \geq C_{f_i}(|\phi_2\rangle)$  and  $C_{f_j}(|\phi_1\rangle) < C_{f_j}(|\phi_2\rangle)$  for  $i \neq j \in \{2, 3\}$ .

For the proofs of Case 2 and Case 3 are similar to that of Case 1. We omit it here.

Combining Cases 1-3, the proof of the theorem is finished.  $\square$

**Remark 4.** Note that, for any irrational number  $k$ , there always exist two series of rational numbers  $\{\mu_n, \nu_n\}_{n=1}^{\infty} \subset (0, 1]$  with  $\mu_n \leq k \leq \nu_n$  for each  $n$  such that  $\lim_{n \rightarrow \infty} \mu_n = \lim_{n \rightarrow \infty} \nu_n = k$ . Thus, if necessary, by making some slight modifications in the proof of Theorem 2, it is enough to require that  $\sum_{j=1}^m p_j C_{g_{l,k}}(|\phi_j\rangle) \geq \sum_{i=1}^n q_i C_{g_{l,k}}(|\psi_i\rangle)$  ( $l = 2, \dots, d$ ) for all rational numbers  $k \in (0, 1]$ .

Finally, we discuss the transformation between any mixed states. For any mixed state  $\rho$ , we call an ensemble  $\{p_j, |\phi_j\rangle\}$  of  $\rho$  optimal if it attains the minimum in Eq.(1), that is,  $C_{g_{l,k}}(\rho) = \sum_j p_j C_{g_{l,k}}(|\phi_j\rangle)$  for  $l = 2, \dots, d$  and  $k \in (0, 1]$ . If such optimal ensemble for any mixed states exists, we can give the definition of coherence convertibility between mixed states by ICO.

**Definition 5.**  $\rho \xrightarrow{\text{ICO}} \sigma$  iff  $\{p_j, |\phi_j\rangle\} \xrightarrow{\text{ICO}} \{q_i, |\psi_i\rangle\}$ . Here,  $\{p_j, |\phi_j\rangle\}$  and  $\{q_i, |\psi_i\rangle\}$  are two optimal ensembles of  $\rho$  and  $\sigma$ , respectively.

**Remark 6.** In the case of both  $\rho$  and  $\sigma$  are pure states, the above definition is equivalent that there is an ICO  $\Phi$  such that  $\Phi(\rho) = \sigma$ . But in other cases, things are not so. For example, assume that  $\rho$  is pure,  $\sigma$  is mixed and there is an ICO  $\Phi$  such that  $\Phi(\rho) = \sigma$ . While the  $\Phi$

corresponds an ensemble  $\{p_i, |\psi_i\rangle\}$  such that  $\rho \xrightarrow{\text{ICO}} \{q_i, |\psi_i\rangle\}$ , the key lies in  $\{p_i, |\psi_i\rangle\}$  may not be an optimal ensemble.

However, we do not know whether an optimal ensemble for any mixed state exists by now. If there is such ensemble, by Theorem 2 and Definition 5, the following result is immediate.

**Theorem 7.** *Assume that  $\rho, \sigma \in \mathcal{S}(H)$  are any two mixed states. Then  $\rho \xrightarrow{\text{ICO}} \sigma$  if and only if  $C_{g_{l,k}}(\rho) \geq C_{g_{l,k}}(\sigma)$  for  $l = 2, \dots, d$  and  $k \in (0, 1]$ .*

#### 4. CONCLUSION

In summary, we find a necessary and sufficient condition for the existence of transformations that converts an ensemble into another ensemble. Different from pure states, for determining such transformation, infinite countable number of conditions based on coherence measures are required. We also point out that, if there exists an optimal ensemble for each mixed state, then the necessary and sufficient conditions for the existence of coherence convertibility between any two mixed states can be obtained. So we partially answer the question raised by Baumgratz et al.. We believe that our results will be fruitful in further developments on convertibility of mixed coherent states.

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